

Simplicities of VOAs Associated to Jordan Algebras of Type B and Character Formulas for Simple Quotients

Hongbo Zhao

1 Introduction

Let V be a $\mathbb{Z}_{\geq 0}$ graded vertex operator algebra (VOA), with $V_0 = \mathbb{C}1$, $V_1 = \{0\}$. Then V_2 has a commutative (but not necessarily associative) algebra structure with operation $a \circ b = a(1)b$. This algebra V_2 is called Griess algebra of V . In [Lam96] and [Lam99], Lam constructed vertex algebras whose Griess algebras are simple Jordan algebras of type A, B, C . For type D simple Jordan algebras the construction was given by Ashihara in [Ash11]; In [AM09] Ashihara and Miyamoto constructed a family of vertex algebras $V_{\mathcal{J},r}$ parametrized by $r \in \mathbb{C}$, whose Griess algebras are isomorphic to type B Jordan algebras \mathcal{J} . The VOA $V_{\mathcal{J},r}$ is further studied by Niibori and Sagaki in [NS10].

One of the main results in [NS10] claims that if \mathcal{J} is not the Jordan algebra of 1×1 matrix, then $V_{\mathcal{J},r}$ is simple if and only if $r \notin \mathbb{Z}$. This suggests that $r \in \mathbb{Z}$ are special and may deserve further study. We show that the simple quotients $\bar{V}_{\mathcal{J},r}$, $r \in \mathbb{Z}_{\neq 0}$ can be constructed by a dual-pair type construction. We also apply the construction to compute the character $\text{Tr}_{|\bar{V}_{\mathcal{J},r}} q^{L(0)}$, $r = -2n$, $n \geq 1$. The Clebsch-Gordan coefficients appear naturally in the character formula.

We give more details about this paper. All vector spaces and Lie groups are assumed to be over \mathbb{C} . Let $(\mathfrak{h}, (\cdot, \cdot))$ be a finite dimensional vector space with a non-degenerate symmetric bilinear form (\cdot, \cdot) , $\dim(\mathfrak{h}) = d$. Then $\mathfrak{h} \otimes \mathfrak{h}$ has an associative algebra structure:

$$(a \otimes b)(u \otimes v) = (b, u)a \otimes v,$$

which induces a Jordan algebra structure on $\mathfrak{h} \otimes \mathfrak{h}$:

$$x \circ y = \frac{1}{2}(xy + yx), \quad \forall x, y \in \mathfrak{h} \otimes \mathfrak{h}.$$

Let \mathcal{J} be the Jordan subalgebra of $\mathfrak{h} \otimes \mathfrak{h}$ consists of symmetric tensors:

$$\mathcal{J} \stackrel{\text{def.}}{=} \text{span}\{L_{a,b}|a, b \in \mathfrak{h}\}, \quad L_{a,b} \stackrel{\text{def.}}{=} a \otimes b + b \otimes a.$$

Then \mathcal{J} is the type B simple Jordan algebra of rank d [JJ49].

Throughout this paper we assume that $d \geq 2$ unless otherwise stated. Let $V_{\mathcal{J},r}$ be the VOA constructed in [AM09] and $\bar{V}_{\mathcal{J},r}$ be the corresponding simple quotient. In [NS10] it is shown that $V_{\mathcal{J},r} = \bar{V}_{\mathcal{J},r}$ if and only if $r \notin \mathbb{Z}$. Our results further show that we can construct $\bar{V}_{\mathcal{J},r}, r \in \mathbb{Z}_{\neq 0}$ explicitly. We divide our constructions into three cases:

Case 1, $r = m, m \geq 1$: Let $(V_m, (\cdot, \cdot))$ be a m -dimensional vector space with a non-degenerate symmetric bilinear form. The tensor product space $\mathfrak{h} \otimes V_m$ is a dm -dimensional vector space with the non-degenerate symmetric bilinear form:

$$((a \otimes u), (b \otimes v)) = (a, b)(u, v).$$

Let $\mathcal{H}(\mathfrak{h} \otimes V_m)$ be the Heisenberg VOA associated to the vector space $\mathfrak{h} \otimes W_m$ [FLM89]. The group $O(m)$ acts on the component V_m , therefore it acts as automorphism on $\mathcal{H}(\mathfrak{h} \otimes V_m)$. We construct $\bar{V}_{\mathcal{J},m}$ as:

$$\bar{V}_{\mathcal{J},m} \stackrel{def.}{=} \mathcal{H}(\mathfrak{h} \otimes V_m)^{O(m)}.$$

Case 2, $r = -2n, n \geq 1$: Let $(W_n, \langle \cdot, \cdot \rangle)$ be a $2n$ -dimensional symplectic space. The tensor product spaces $\mathfrak{h} \otimes W_n$ is a $2dn$ -dimensional symplectic space with the symplectic form:

$$\langle (a \otimes u), (b \otimes v) \rangle = (a, b)\langle u, v \rangle.$$

Let $\mathcal{A}(\mathfrak{h} \otimes W_n)$ be the ‘symplectic Fermion’ super vertex operator algebra(SVOA) associated to the vector space $\mathfrak{h} \otimes W_n$ [Kau95]. The group $Sp(2n)$ acts on the component W_n , therefore it acts as automorphism on $\mathcal{A}(\mathfrak{h} \otimes W_n)$. We construct $\bar{V}_{\mathcal{J},-2n}$ as:

$$\bar{V}_{\mathcal{J},-2n} \stackrel{def.}{=} \mathcal{A}(\mathfrak{h} \otimes W_n)^{Sp(2n)}.$$

Case 3, $r = -2n+1, n \geq 1$: Let $(W, (\cdot, \cdot))$ be an orthosymplectic superspace with $\text{sdim}(W) = (1|2n)$. The tensor product space $\mathfrak{h} \otimes W$ is an orthosymplectic superspace with the supersymmetric bilinear form:

$$((a \otimes u), (b \otimes v)) = (a, b)(u, v),$$

and $\text{sdim}(\mathfrak{h} \otimes W) = (d|2nd)$. Let

$$\mathcal{H}(\mathfrak{h} \otimes V_1) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n) \simeq \mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n)$$

be the SVOA associated to the superspace $\mathfrak{h} \otimes W$. The ‘supergroup’ $Osp(1|2n)$ which means the pair $(\mathfrak{osp}(1|2n), O(1) \times Sp(2n))$ here(See for example, [Ser01], [DKW⁺99]), acts on the component W , therefore it acts on $\mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n)$. We construct $\bar{V}_{\mathcal{J},-2n+1}, n \geq 1$ as:

$$\bar{V}_{\mathcal{J},-2n+1} \stackrel{def.}{=} (\mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n))^{Osp(1|2n)}.$$

We have the following theorem:

Theorem 1. *The VOA $\bar{V}_{\mathcal{J},r}(d \geq 2, r \in \mathbb{Z}_{\neq 0})$ satisfies following properties:*

- (1). $(\bar{V}_{\mathcal{J},r})_0 = \mathbb{C}1, (\bar{V}_{\mathcal{J},r})_1 = \{0\}$, and the central charge equals to dr . The Griess algebra $(\bar{V}_{\mathcal{J},r})_2$ is isomorphic to the Jordan algebra \mathcal{J} .
- (2). $\bar{V}_{\mathcal{J},r}$ is generated by $(\bar{V}_{\mathcal{J},r})_2$.
- (3). $\bar{V}_{\mathcal{J},r}$ is the simple quotient of $V_{\mathcal{J},r}$.

By using Case 2 of the construction for $\bar{V}_{\mathcal{J},r}$, we give character formulas for $\bar{V}_{\mathcal{J},r}, r = -2n, n \geq 1$. We recall some facts about $\mathfrak{osp}(1|2n)$ (See for example, [Kac77]), $\mathfrak{sp}(2n)$, and $\mathfrak{so}(2n+1)$. Let Φ_s be the root system of $\mathfrak{osp}(1|2n)$ with even roots Φ_0 and odd roots Φ_1 :

$$\begin{aligned}\Phi_0 &= \{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j, 2\epsilon_i, -2\epsilon_i | i \neq j, 1 \leq i, j \leq n\}, \\ \Phi_1 &= \{\epsilon_i, -\epsilon_i | 1 \leq i \leq n\}, \quad \Phi_s = \Phi_0 \cup \Phi_1.\end{aligned}$$

then Φ_0 is the root system of $\mathfrak{sp}(2n)$, which is an even subalgebra of $\mathfrak{osp}(1|2n)$. A choice of positive roots is:

$$\Phi_0^+ = \{-\epsilon_i + \epsilon_j, \epsilon_i + \epsilon_j, 2\epsilon_i | 1 \leq i < j \leq n\}, \quad \Phi_1^+ = \{\epsilon_i | 1 \leq i \leq n\}.$$

We note that the root system Φ of $\mathfrak{so}(2n+1)$ can be viewed as a sub root system of Φ_s :

$$\Phi = \{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j, \epsilon_i, -\epsilon_i | i \neq j, 1 \leq i, j \leq n\},$$

with positive roots $\Phi^+ \subseteq \Phi_0^+ \cup \Phi_1^+$:

$$\Phi^+ = \{-\epsilon_i + \epsilon_j, \epsilon_i + \epsilon_j, \epsilon_i | 1 \leq i < j \leq n\}.$$

Introduce elements

$$\rho_0 \stackrel{\text{def.}}{=} \frac{1}{2} \sum_{\alpha \in \Phi_0} \alpha, \quad \rho_1 \stackrel{\text{def.}}{=} \frac{1}{2} \sum_{\alpha \in \Phi_1} \alpha.$$

Let Λ_+^0 be the set of dominant integral weights of $\mathfrak{sp}(2n)$, $L(\lambda)$ be the simple $\mathfrak{sp}(2n)$ -module with highest weight $\lambda \in \Lambda_+^0$. Let $m_{\lambda_1, \dots, \lambda_d}^\mu$ denote the dimension of the ‘multiplicity space’:

$$m_{\lambda_1, \dots, \lambda_d}^\mu \stackrel{\text{def.}}{=} \dim(\text{Hom}_{\mathfrak{sp}(2n)}(L(\mu), L(\lambda_1) \otimes \dots \otimes L(\lambda_d))), \quad \mu, \lambda_i \in \Lambda_+^0.$$

Then we have:

Theorem 2. *Let $P(q)$ be the generating function of integer partitions:*

$$P(q) \stackrel{\text{def.}}{=} \prod_{j \geq 1} (1 - q^j)^{-1}.$$

Define the ‘branching function’ $B_\lambda(q)$:

$$B_\lambda(q) \stackrel{\text{def.}}{=} q^{\frac{1}{2}(\lambda + \rho_1, \lambda + \rho_1) - \frac{1}{2}(\rho_1, \rho_1)} P(q)^n \prod_{\alpha \in \Phi^+} (1 - q^{(\lambda + \rho_0, \alpha)}). \quad (1)$$

Then:

$$\mathrm{Tr}|_{\bar{V}_{\mathcal{J},r}} q^{L(0)} = \sum_{\lambda_1, \dots, \lambda_d \in \Lambda_+^0} m_{\lambda_1, \dots, \lambda_d}^0 B_{\lambda_1}(q) \cdots B_{\lambda_d}(q).$$

This paper is organized as follows. In Section 2 we briefly review the construction of $V_{\mathcal{J},r}$ in [AM09] and main results of [NS10]. We reprove $V_{\mathcal{J},r} = \bar{V}_{\mathcal{J},r}$ if $r \notin \mathbb{Z}$ using a different method. We give the constructions of Case 1 and Case 2 in Section 3. Then the construction of Case 3 is given in Section 4. We prove Theorem 1 in Section 5 and compute character formulas for Case 2 in Section 6. As by-products we can also give natural explanations to some main results in [NS10].

Acknowledgements I would like to thank professor Y. Zhu for discussions.

2 The VOA $V_{\mathcal{J},r}$ and Its Simplicity When $r \notin \mathbb{Z}$

We first review the construction of Ashihara and Miyamoto in [AM09], and main results of Niibori and Sagaki in [NS10]. Let \mathfrak{h} be a finite dimensional vector space with a symmetric non-degenerate bilinear form (\cdot, \cdot) , $\dim(\mathfrak{h}) = d$. The Heisenberg Lie algebra associated to $(\mathfrak{h}, (\cdot, \cdot))$ is:

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

with the Lie bracket:

$$[a(m), b(n)] = m(a, b)\delta_{m+n,0}c, \quad [x, c] = 0, \quad \forall x \in \hat{\mathfrak{h}}.$$

Here $a(m) = at^m \in \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$. It is well known that

$$\hat{\mathfrak{h}}_- \stackrel{def.}{=} \mathfrak{h} \otimes \mathbb{C}t^{-1}[t^{-1}]$$

is a commutative Lie subalgebra. The Fock space $S(\hat{\mathfrak{h}}_-) \simeq U(\hat{\mathfrak{h}}_-) \cdot 1$ has a vertex operator algebra structure [FLM89]. Denote this VOA by $\mathcal{H}(\mathfrak{h})$.

It is easy to check that $U(\hat{\mathfrak{h}})$ is closed under the ‘new’ Lie bracket $[x, y]_{new}$:

$$[x, y]_{new} \stackrel{def.}{=} \frac{1}{c}[x, y], \quad \forall x, y \in U(\hat{\mathfrak{h}}).$$

And the subspace \mathcal{L} :

$$\mathcal{L} \stackrel{def.}{=} \mathrm{span}\{a(m)b(n) \mid a, b \in \mathfrak{h}, m, n \in \mathbb{Z}\}$$

is a Lie subalgebra.

Define the ‘normal ordering’:

$$: a(m)b(n) := \begin{cases} b(n)a(m), & m \geq n, \\ a(m)b(n), & m < n. \end{cases}$$

Set

$$L_{a,b}(m, n) \stackrel{def.}{=} \frac{1}{2} : a(m)b(n) :$$

and define a function

$$\mathbb{1}_m = \begin{cases} 1, & m \geq 0, \\ 0, & m < 0. \end{cases}$$

By a direct computation it is easy to show that for $L_{a,b}(m, n), L_{u,v}(k, l) \in \mathcal{L}$:

$$\begin{aligned} & [L_{a,b}(m, n), L_{u,v}(k, l)]_{new} \\ &= \frac{1}{2} n \delta_{n+k}(b, u) L_{a,v}(m, l) + \frac{1}{2} m \delta_{m+k}(a, u) L_{b,v}(n, l) \\ &+ \frac{1}{2} n \delta_{n+l}(b, v) L_{a,u}(m, k) + \frac{1}{2} m \delta_{m+l}(a, v) L_{b,u}(n, k) \\ &+ \frac{c}{2} n m \delta_{n+k} \delta_{m+l}(b, u)(a, v) \mathbb{1}_{m-l} + \frac{c}{2} m n \delta_{m+k} \delta_{n+l}(a, u)(b, v) \mathbb{1}_{n-l} \\ &+ \frac{c}{2} m n \delta_{n+l} \delta_{m+k}(b, v)(a, u) \mathbb{1}_{m-k} + \frac{c}{2} m n \delta_{m+l} \delta_{n+k}(a, v)(b, u) \mathbb{1}_{n-k}. \end{aligned} \quad (2)$$

Let

$$\begin{aligned} \mathfrak{B}_+ &\stackrel{def.}{=} \text{span}\{L_{a,b}(m, n) | n \geq 0 \text{ or } m \geq 0\}, \\ \mathcal{L}_- &\stackrel{def.}{=} \text{span}\{L_{a,b}(m, n) | m, n < 0\}, \quad \mathcal{L}_+ \stackrel{def.}{=} \mathfrak{B}_+ \oplus \mathbb{C}c. \end{aligned}$$

Then we have a decomposition of \mathcal{L} :

$$\mathcal{L} = \mathcal{L}_- \oplus \mathcal{L}_+ = \mathcal{L}_- \oplus \mathfrak{B}_+ \oplus \mathbb{C}c.$$

Define a 1-dimensional \mathcal{L}_+ -module $\mathbb{C}1$:

$$x1 = 0, \quad \forall x \in \mathfrak{B}_+, \quad c1 = r1.$$

Then by induction from $U(\mathcal{L}_+)$ to $U(\mathcal{L})$, we have a $U(\mathcal{L})$ -module M_r :

$$\begin{aligned} M_r &\stackrel{def.}{=} U(\mathcal{L}) \otimes_{U(\mathcal{L}_+)} \mathbb{C}1 \cong U(\mathcal{L}_-)1 \\ &= \text{span}\{L_{a_1, b_1}(-m_1, -n_1) \cdots L_{a_k, b_k}(-m_k, -n_k) \cdot 1 | \\ &\quad m_i, n_i \in \mathbb{Z}_{\geq 1}, a_i, b_i \in \mathfrak{h}\}. \end{aligned} \quad (3)$$

Because c acts as r on M_r so we can take $c = r$ in (2).

For $a, b \in \mathfrak{h}$ define operators $L_{a,b}(l)$ and fields $L_{a,b}(z)$:

$$L_{a,b}(l) \stackrel{def.}{=} \sum_{k \in \mathbb{Z}} L_{a,b}(-k + l - 1, k), \quad L_{a,b}(z) \stackrel{def.}{=} \sum_{l \in \mathbb{Z}} L_{a,b}(l) z^{-l-1}.$$

It is proved in [AM09] that these fields are mutually local.

So these mutually local fields generate a vertex algebra (See for example, [Kac98]), denoted by $V_{\mathcal{J},r}$:

$$V_{\mathcal{J},r} = \text{span}\{L_{a_1,b_1}(m_1) \cdots L_{a_k,b_k}(m_k) \cdot 1 \mid m_i \in \mathbb{Z}, a_i, b_i \in \mathfrak{h}\}.$$

The first main result in [NS10] claims that $M_r = V_{\mathcal{J},r}$ holds.

Let $\{e_1, \dots, e_d\}$ be an orthonormal basis of \mathfrak{h} . Then the Virasoro element ω is given by:

$$\omega = \sum_k L_{e_k, e_k}(-1, -1) \cdot 1.$$

$L(0) = \omega(1)$ gives a gradation on $V_{\mathcal{J},r}$:

$$V_{\mathcal{J},r} = \bigoplus_{k \geq 0} (V_{\mathcal{J},r})_k.$$

It is easy to see that

$$(V_{\mathcal{J},r})_0 = \mathbb{C}1, \quad (V_{\mathcal{J},r})_1 = \{0\},$$

and the Griess algebra $(V_{\mathcal{J},r})_2$ is isomorphic to \mathcal{J} :

$$L_{a,b}(-1, -1) \cdot 1 \mapsto L_{a,b} \stackrel{\text{def.}}{=} a \otimes b + b \otimes a.$$

We observe that the Lie algebra \mathcal{L} is closely related to the infinite rank symplectic Lie algebra C_∞ (See for example, chap. 7 of [Kac94].) The Lie algebra C_∞ is a subalgebra of \mathcal{L} , an ideal $\mathcal{I} \subseteq \mathcal{L}$ acts as 0 on $V_{\mathcal{J},r}$, and $C_\infty \oplus \mathbb{C}c$ is isomorphic to \mathcal{L}/\mathcal{I} . We also observe that $V_{\mathcal{J},r}$ is a generalized Verma module for C_∞ [KR93]. So we can reprove $V_{\mathcal{J},r} = \bar{V}_{\mathcal{J},r}$ if $r \notin \mathbb{Z}$, by using the irreducibility criteria for the generalized Verma module.

It is easy to see that

$$W_N \stackrel{\text{def.}}{=} \text{span}\{a(i) \mid a \in \mathfrak{h}, 1 \leq |i| \leq N\}$$

is a $2dN$ -dimensional symplectic space. The symplectic form is given by:

$$\langle a(m), b(n) \rangle = [a(m), b(n)]_{\text{new}} = m(a, b) \delta_{m+n, 0}.$$

Suppose for $k \in \mathbb{Z}$ such that $1 \leq k \leq dN$, $k = (i-1)N + j$, $1 \leq i \leq d$, $1 \leq j \leq N$, set

$$v_k = \frac{1}{j} e_i(j), \quad v_{-k} = e_i(-j).$$

It is easy to check that $\{v_k \mid 1 \leq |k| \leq dN\}$ is a symplectic basis of W_N such that $\langle v_k, v_l \rangle = \delta_{k+l, 0}$, $\forall k > 0$.

We need the following lemma:

Lemma 1.

$$\text{span}\left\{\frac{1}{2}(v_k v_l + v_l v_k) \mid 1 \leq |k|, |l| \leq dN\right\}$$

is a Lie algebra isomorphic to $\mathfrak{sp}(2dn)$.

Proof: It is easy to see that the adjoint action of $x \in \text{span}\{\frac{1}{2}(v_k v_l + v_l v_k) | 1 \leq |k|, |l| \leq dN\}$ on W_N

$$x \cdot v \stackrel{def.}{=} [x, v]_{new}$$

preserves the symplectic form on W_N :

$$\langle x \cdot u, v \rangle + \langle u, x \cdot v \rangle = 0, \quad \forall u, v \in W_N.$$

So

$$\text{span}\{\frac{1}{2}(v_k v_l + v_l v_k) | 1 \leq |k|, |l| \leq dN\} \subseteq \mathfrak{sp}(2dn).$$

We conclude the proof of Lemma 1 by counting the dimension.

For convenience we set:

$$\mathfrak{g}^{(N)} \stackrel{def.}{=} \mathfrak{sp}(2dn) \simeq \text{span}\{\frac{1}{2}(v_k v_l + v_l v_k) | 1 \leq |k|, |l| \leq dN\}.$$

We now analyze the root space decomposition of $\mathfrak{g}^{(N)}$. Note that:

$$\mathfrak{g}^{(N)} = \mathfrak{g}_+^{(N)} \oplus \mathfrak{h}^{(N)} \oplus \mathfrak{g}_-^{(N)},$$

where

$$\begin{aligned} \mathfrak{g}_+^{(N)} &= \text{span}\{\frac{1}{2}(v_k v_l + v_l v_k) | k + l > 0\} = \text{span}\{v_k v_l | k + l > 0\}, \\ \mathfrak{h}^{(N)} &= \text{span}\{\frac{1}{2}(v_k v_l + v_l v_k) | k + l = 0\} = \text{span}\{v_{-k} v_k + \frac{c}{2} | k > 0\}, \\ \mathfrak{g}_-^{(N)} &= \text{span}\{\frac{1}{2}(v_k v_l + v_l v_k) | k + l < 0\} = \text{span}\{v_k v_l | k + l < 0\}. \end{aligned}$$

Introduce elements $\epsilon_k \in (\mathfrak{h}^{(N)})^*, k = 1, \dots, dN$ such that:

$$\epsilon_l(v_{-k} v_k + \frac{c}{2}) = -\delta_{k,l}.$$

The positive and negative roots with respect to the triangular decomposition are:

$$\begin{aligned} \Phi_+^{(N)} &= \{+\epsilon_i + \epsilon_j | i \leq j\} \cup \{-\epsilon_i + \epsilon_j | i < j\}, \\ \Phi_-^{(N)} &= \{-\epsilon_i - \epsilon_j | i \leq j\} \cup \{+\epsilon_i - \epsilon_j | i < j\}. \end{aligned}$$

The corresponding simple roots are:

$$\Delta^{(N)} = \{2\epsilon_1\} \cup \{-\epsilon_i + \epsilon_{i+1} | 1 \leq i < dN\}.$$

The half sum of positive roots is:

$$\rho^{(N)} = \frac{1}{2} \sum_{\alpha \in \Phi_+^{(N)}} \alpha = \sum_{1 \leq i \leq dN} i \epsilon_i.$$

We recall ‘generalized Verma module of scalar type’ for $\mathfrak{g}^{(N)}$ (For notations and conventions, see for example, chapter 9 of [Hum08]). Define:

$$\begin{aligned}\mathfrak{n}_-^{(N)} &\stackrel{def.}{=} \text{span}\{v_k v_l \mid k, l < 0\}, \\ \mathfrak{l}^{(N)} &\stackrel{def.}{=} \text{span}\{v_k v_l + \frac{c}{2} \delta_{k+l,0} \mid k < 0, l > 0\}, \quad \mathfrak{u}^{(N)} \stackrel{def.}{=} \text{span}\{v_k v_l \mid k, l > 0\}, \\ \mathfrak{p}^{(N)} &\stackrel{def.}{=} \mathfrak{l}^{(N)} \oplus \mathfrak{u}^{(N)}.\end{aligned}$$

Then we have decompositions:

$$\mathfrak{g}^{(N)} = \mathfrak{p}^{(N)} \oplus \mathfrak{n}_-^{(N)} = \mathfrak{l}^{(N)} \oplus \mathfrak{u}^{(N)} \oplus \mathfrak{n}_-^{(N)}.$$

And we define the following set $\Phi_I^{(N)}$:

$$\Phi_I^{(N)} \stackrel{def.}{=} \{-\epsilon_i + \epsilon_j \mid i < j\} \cup \{\epsilon_i - \epsilon_j \mid i < j\}.$$

Then $\mathfrak{l}^{(N)}$ is spanned by $\mathfrak{h}^{(N)}$ and root spaces $(\mathfrak{g}^{(N)})_\alpha$, where $\alpha \in \Phi_I^{(N)}$.

Define the 1-dimensional $\mathfrak{p}^{(N)}$ -module of weight $\lambda^{(N)} \in (\mathfrak{h}^{(N)})^*$ spanned by the element 1 such that:

$$x \cdot 1 = 0, \quad h \cdot 1 = \lambda^{(N)}(h) \cdot 1 \quad \forall h \in \mathfrak{h}^{(N)}, x \in (\mathfrak{g}^{(N)})_\alpha, \alpha \in \Phi_I^{(N)} \cup \Phi_+^{(N)}.$$

Then we define the following generalized Verma module $M_I(\lambda^{(N)})$:

$$M_I(\lambda^{(N)}) \stackrel{def.}{=} U(\mathfrak{g}^{(N)}) \otimes_{U(\mathfrak{p}^{(N)})} \mathbb{C} \cdot 1 \simeq U(\mathfrak{n}_-^{(N)}) \cdot 1. \quad (4)$$

It is known that $M_I(\lambda^{(N)})$ is a ‘generalized Verma module of scalar type’ (See for example [Hum08]).

We want to show that $V_{\mathcal{J},r} = M_r$ is a generalized Verma module for C_∞ ([KR93]). Observe that unions of $\mathfrak{g}^{(N)}$ is isomorphic to the infinite rank symplectic Lie algebra C_∞ :

$$C_\infty = \cup_{k \geq 1} \mathfrak{g}^{(k)},$$

Define

$$\mathcal{I} \stackrel{def.}{=} \text{span}\{L_{a,b}(m,n) \mid a, b \in \mathfrak{h}, mn = 0\},$$

Then \mathcal{I} is an ideal of \mathcal{L} , \mathcal{I} acts as 0 on $V_{\mathcal{J},r}$. We also have:

$$\mathcal{L} = C_\infty \bigoplus \mathcal{I} \bigoplus \mathbb{C}c. \quad (5)$$

Note that there are increasing exhaustive filtration:

$$\{0\} = \mathfrak{n}_-^{(0)} \subseteq \mathfrak{n}_-^{(1)} \cdots \subseteq \mathcal{L}_-, \quad \{0\} = \mathfrak{p}^{(0)} \subseteq \mathfrak{p}^{(1)} \cdots \subseteq C_\infty \cap \mathcal{L}_+. \quad (6)$$

It is easy to compute that:

$$(v_{-k}v_k + \frac{c}{2}) \cdot 1 = \frac{r}{2} \cdot 1, \quad v_kv_l \cdot 1 = 0, \quad \forall v_kv_l \in (\mathfrak{g}^{(N)})_\alpha, \alpha \in \Phi_I^{(N)} \cup \Phi_+^{(N)}.$$

So

$$\lambda^{(N)} = -\frac{r}{2} \sum_{i=1, \dots, dN} \epsilon_i.$$

By comparing (3),(4) and use (6), we have an embedding of $\mathfrak{g}^{(N)}$ -module:

$$M_I(\lambda^{(N)}) \hookrightarrow M_r.$$

We also have an exhaustive filtration:

$$\{0\} \subseteq M_I(\lambda^{(1)}) \subseteq \dots \subseteq M_I(\lambda^{(N)}) \subseteq \dots \subseteq M_r. \quad (7)$$

So we conclude that $V_{\mathcal{J},r}$ is a ‘generalized Verma module of scalar type’ for C_∞ .

Proof $V_{\mathcal{J},r} = \bar{V}_{\mathcal{J},r}$ **when** $r \notin \mathbb{Z}$. We first need the following lemma which can be found in Proposition 3.4 and its proof in [NS10]:

Lemma 2 ([NS10]). *All proper VOA ideals of $V_{\mathcal{J},r}$ are also proper \mathcal{L} -submodules of $V_{\mathcal{J},r}$, so $V_{\mathcal{J},r}$ is simple if and only if $M_r = V_{\mathcal{J},r}$ is simple as a \mathcal{L} -module.*

From (5) it is easy to see that \mathcal{L} -submodules of $V_{\mathcal{J},r}$ are also in 1-1 correspondence with the C_∞ submodules of $V_{\mathcal{J},r}$. So the simplicity of the VOA $V_{\mathcal{J},r}$, is reduced to the simplicity of $V_{\mathcal{J},r}$ as C_∞ -module.

We need another lemma on simplicity of finite dimensional generalized Verma module of scalar type which can be found in [Hum08], 9.12, (a) of the Theorem:

Lemma 3. *if $\lambda^{(N)}$ is a dominant integral weight for $\mathfrak{g}^{(N)}$ and*

$$\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle \notin \mathbb{Z}_{>0}, \quad \forall \beta \in \Phi_+^{(N)} - \Phi_I^{(N)}.$$

then $M_I(\lambda^{(N)})$ is an irreducible $\mathfrak{g}^{(N)}$ -module.

It is easy to compute that

$$\beta^\vee = \begin{cases} -v_{-k}v_k - v_{-l}v_l - c, & k \neq l, \\ -v_{-k}v_k - \frac{c}{2}, & k = l, \end{cases}$$

for $\beta = \epsilon_k + \epsilon_l \in \Phi_+^{(N)} - \Phi_I^{(N)} = \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j \leq dN\}$. So

$$\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle = \begin{cases} -r + k + l, & k \neq l, \\ -r + 2k, & k = l. \end{cases}$$

It’s obvious that when $r \notin \mathbb{Z}$,

$$\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle \notin \mathbb{Z}_{>0}, \quad \forall \beta \in \Phi_+^{(N)} - \Phi_I^{(N)}.$$

By Lemma 3 it is shown that $M_I(\lambda^{(N)})$ is irreducible as $\mathfrak{g}^{(N)}$ -module when $r \notin \mathbb{Z}$.

We now conclude the proof by contradiction. Suppose the contrary that $V_{\mathcal{J},r}$ is not simple when $r \notin \mathbb{Z}$, then it has a proper C_∞ submodule M . Note that the filtration (7) is exhaustive, we deduce that $M \cap M_I(\lambda^{(N)})$ is a proper $\mathfrak{g}^{(N)}$ -submodule of $M_I(\lambda^{(N)})$, for some N . This contradicts to the result that $M_I(\lambda^{(N)})$ is irreducible for all N when $r \notin \mathbb{Z}$. So we conclude the proof.

3 Dual Pair Realization of $\bar{V}_{\mathcal{J},r}$, $r = m \geq 1$ or $r = -2n, n \geq 1$

In this section we give detailed constructions of Case 1 and Case 2 in the introduction.

Construction of Case 1, $r = m \geq 1$. Recall that $\mathfrak{h} \otimes V_m$ is a dm -dimensional vector space with a non-degenerate symmetric bilinear form. Then we have the corresponding Lie algebra $\widehat{\mathfrak{h} \otimes V_m}$ and the corresponding Heisenberg VOA $\mathcal{H}(\mathfrak{h} \otimes V_m)$. The group $O(dm)$ acts on $\widehat{\mathfrak{h} \otimes V_m}$ and $\mathcal{H}(\mathfrak{h} \otimes V_m)$ as automorphism. The subgroup $O(m)$ acts on the component V_m , then we construct $\bar{V}_{\mathcal{J},m}$ as fixpoint sub VOA of $\mathcal{H}(\mathfrak{h} \otimes V_m)$:

$$\bar{V}_{\mathcal{J},m} \stackrel{\text{def.}}{=} \mathcal{H}(\mathfrak{h} \otimes V_m)^{O(m)}.$$

We describe $\mathcal{H}(\mathfrak{h} \otimes V_m)^{O(m)}$ more explicitly using the invariant theory for $O(m)$. Define the fixpoint Lie subalgebra \mathcal{L}_m of $\widehat{\mathfrak{h} \otimes V_m}$:

$$\mathcal{L}_m \stackrel{\text{def.}}{=} (\widehat{\mathfrak{h} \otimes V_m})^{O(m)}.$$

Let f_1, \dots, f_m be an orthonormal basis of V_m , and we set

$$L_{a,b}^m(k, l) \stackrel{\text{def.}}{=} \frac{1}{2} \sum_{i=1, \dots, m} : (a \otimes f_i)(k) (b \otimes f_i)(l) : .$$

By the invariant theory for $O(m)$, we have:

$$\mathcal{L}_m = \text{span}\{L_{a,b}^m(k, l), c \mid a, b \in \mathfrak{h}, k, l \in \mathbb{Z}\}.$$

A direct computation shows that:

$$\begin{aligned} & [L_{a,b}^m(s, t), L_{u,v}^m(k, l)] \\ &= \frac{c}{2} t \delta_{t+k}(b, u) L_{a,v}^m(s, l) + \frac{c}{2} s \delta_{s+k}(a, u) L_{b,v}^m(t, l) \\ &+ \frac{c}{2} t \delta_{t+l}(b, v) L_{a,u}^m(s, k) + \frac{c}{2} s \delta_{s+l}(a, v) L_{b,u}^m(t, k) \\ &+ \frac{mc^2}{2} s t \delta_{t+k} \delta_{s+l}(b, u)(a, v) \mathbb{1}_{s-l} + \frac{mc^2}{2} s t \delta_{s+k} \delta_{t+l}(a, u)(b, v) \mathbb{1}_{t-l} \end{aligned}$$

$$+\frac{mc^2}{2}st\delta_{t+l}\delta_{s+k}(b,v)(a,u)\mathbb{1}_{s-k}+\frac{mc^2}{2}st\delta_{s+l}\delta_{t+k}(a,v)(b,u)\mathbb{1}_{t-k}. \quad (8)$$

Because c acts as 1 on $\mathcal{H}(\mathfrak{h} \otimes V_m)$ so we can take $c = 1$ in (8). We remark that the computation of (8) is very similar to (2) because

$$[(a \otimes f_i)(k)(b \otimes f_i)(l), (a \otimes f_j)(k)(b \otimes f_j)(l)] = 0, \forall i \neq j.$$

And we also have the following description of $\bar{V}_{\mathcal{J},m}$ by the invariant theory for $O(m)$:

$$\begin{aligned} \bar{V}_{\mathcal{J},m} &= \mathcal{H}(\mathfrak{h} \otimes V_m)^{O(m)} \simeq (S(\widehat{(\mathfrak{h} \otimes V_m)_-}) \cdot 1)^{O(m)} \\ &= \text{span}\{L_{a_1, b_1}^m(-m_1, -n_1) \cdots L_{a_k, b_k}^m(-m_k, -n_k) \cdot 1 \mid a_i, b_i \in \mathfrak{h}, m_i, n_i \geq 1\}. \end{aligned}$$

The Virasoro element ω is given by

$$\omega = \sum_{k=1, \dots, d} L_{e_k, e_k}^m.$$

It is computed that

$$\omega(3)\omega = \frac{dm}{2}\omega,$$

So the central charge equals to dm .

Construction of Case 2, $r = -2n, n \geq 1$. Recall that $\mathfrak{h} \otimes W_n$ is a $2dn$ -dimensional symplectic space. We first recall the construction of SVOA $\mathcal{A}(\mathfrak{h} \otimes W_n)$. Define the Lie super algebra $\widehat{\mathfrak{h} \otimes W_n}$:

$$\widehat{\mathfrak{h} \otimes W_n} = (\mathfrak{h} \otimes W_n) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

with the odd subspace

$$(\mathfrak{h} \otimes W_n) \otimes \mathbb{C}[t, t^{-1}]$$

and the even subspace spanned by c . The the Lie super bracket is given by:

$$[a(m), b(n)] = m\langle a, b \rangle \delta_{m+n, 0} c, \quad [x, c] = 0, \quad \forall x \in \widehat{\mathfrak{h} \otimes W_n}, \quad a, b \in \mathfrak{h} \otimes W_n.$$

Here $a(m) = at^m$. It is easy to check that

$$\widehat{\mathfrak{h} \otimes W_n} \stackrel{\text{def.}}{=} (\mathfrak{h} \otimes W_n) \otimes \mathbb{C}t^{-1}[t^{-1}]$$

is a super-commutative Lie subalgebra. The Fermionic Fock space $\bigwedge (\widehat{\mathfrak{h} \otimes W_n})_- \cdot 1$ has a SVOA structure, and the field associated to $x = x(-1) \cdot 1$ is:

$$Y(x, z) = \sum_k x(k)z^{-k-1}, \quad x \in \mathfrak{h} \otimes W_n.$$

This is called ‘symplectic Fermion’ SVOA in the literatures [Kau95], denoted by $\mathcal{A}(\mathfrak{h} \otimes W_n)$.

It's obvious that $Sp(2dn)$ acts on $\widehat{\mathfrak{h} \otimes W_n}$ and $\mathcal{A}(\mathfrak{h} \otimes W_n)$ as automorphism. The subgroup $Sp(2n)$ acts on the component W_n . We construct $\bar{V}_{\mathcal{J}, -2n}$ as the fixpoint sub SVOA of $\mathcal{A}(\mathfrak{h} \otimes W_n)$:

$$\bar{V}_{\mathcal{J}, -2n} \stackrel{def.}{=} \mathcal{A}(\mathfrak{h} \otimes W_n)^{Sp(2n)}.$$

We also describe $\bar{V}_{\mathcal{J}, -2n}$ explicitly using the invariant theory for $Sp(2n)$. Define the fixpoint Lie sub-superalgebra:

$$\mathcal{L}_{-2n} \stackrel{def.}{=} (\widehat{\mathfrak{h} \otimes W_n})^{Sp(2n)}.$$

Let $\psi_1, \dots, \psi_n, \psi_1^*, \dots, \psi_n^*$ be a symplectic basis of W_n such that

$$\langle \psi_i^*, \psi_j \rangle = \delta_{i,j}, \langle \psi_i^*, \psi_j^* \rangle = \langle \psi_i, \psi_j \rangle = 0.$$

Define the ‘Fermionic normal ordering’ : $a(m)b(n)$:

$$: a(m)b(n) := \begin{cases} (-1)^{p(a)p(b)} b(n)a(m), & m \geq n, \\ a(m)b(n), & m < n. \end{cases}$$

where $p(\cdot)$ is the parity function. Set

$$L_{a,b}^{-2n}(k,l) \stackrel{def.}{=} \frac{1}{2} \sum_{j=1, \dots, n} : (a \otimes \psi_j)(k)(b \otimes \psi_j^*)(l) : - \frac{1}{2} \sum_{j=1, \dots, n} : (a \otimes \psi_j^*)(k)(b \otimes \psi_j)(l) : .$$

By the invariant theory for $Sp(2n)$, we have:

$$\mathcal{L}_{-2n} = \text{span}\{L_{a,b}^{-2n}(k,l), c \mid a, b \in \mathfrak{h}, k, l \in \mathbb{Z}\}.$$

Note that \mathcal{L}_{-2n} and $\bar{V}_{\mathcal{J}, -2n}$ are both even so we can drop the adjective ‘super’. A direct computation shows that:

$$\begin{aligned} & [L_{a,b}^{-2n}(s,t), L_{u,v}^{-2n}(k,l)] \\ &= \frac{c}{2} t \delta_{t+k}(b,u) L_{a,v}^{-2n}(s,l) + \frac{c}{2} s \delta_{s+k}(a,u) L_{b,v}^{-2n}(t,l) \\ &+ \frac{c}{2} t \delta_{t+l}(b,v) L_{a,u}^{-2n}(s,k) + \frac{c}{2} s \delta_{s+l}(a,v) L_{b,u}^{-2n}(t,k) \\ &- nc^2 st \delta_{t+k} \delta_{s+l}(b,u)(a,v) \mathbb{1}_{s-l} - nc^2 st \delta_{s+k} \delta_{t+l}(a,u)(b,v) \mathbb{1}_{t-l} \\ &- nc^2 st \delta_{t+l} \delta_{s+k}(b,v)(a,u) \mathbb{1}_{s-k} - nc^2 st \delta_{s+l} \delta_{t+k}(a,v)(b,u) \mathbb{1}_{t-k}. \end{aligned} \quad (9)$$

Because c acts as 1 on $\mathcal{A}(\mathfrak{h} \otimes W_n)$, therefore we can take $c = 1$ in (9). By the invariant theory for $Sp(2n)$ the fixpoint Sub VOA $\bar{V}_{\mathcal{J}, -2n}$ is explicitly described by:

$$\bar{V}_{\mathcal{J}, -2n} = \mathcal{A}(\mathfrak{h} \otimes W_n)^{Sp(2n)} \simeq (\bigwedge (\widehat{\mathfrak{h} \otimes W_n})_- \cdot 1)^{Sp(2n)}$$

$$=\text{span}\{L_{a_1,b_1}^{-2n}(-m_1,-n_1)\cdots L_{a_k,b_k}^{-2n}(-m_k,-n_k)\cdot 1 \mid a_i,b_i \in \mathfrak{h}, m_i, n_i \geq 1\}.$$

The Virasoro element is:

$$\omega = \sum_{k=1,\dots,d} L_{e_k,e_k}^{-2n}.$$

It is computed that

$$\omega(3)\omega = -dn\omega,$$

So the central charge equals to $-2dn$.

We now compare (2), (8), and (9). The main observation of our construction is that we can unify (8) and (9) when $r = m \geq 1$ or $r = -2n, n \geq 1$:

$$\begin{aligned} & [L_{a,b}^r(s,t), L_{u,v}^r(k,l)] \\ &= \frac{1}{2}t\delta_{t+k}(b,u)L_{a,v}^r(s,l) + \frac{1}{2}s\delta_{s+k}(a,u)L_{b,v}^r(t,l) \\ &+ \frac{1}{2}t\delta_{t+l}(b,v)L_{a,u}^r(s,k) + \frac{1}{2}s\delta_{s+l}(a,v)L_{b,u}^r(t,k) \\ &+ \frac{r}{2}st\delta_{t+k}\delta_{s+l}(b,u)(a,v)\mathbb{1}_{s-l} + \frac{r}{2}st\delta_{s+k}\delta_{t+l}(a,u)(b,v)\mathbb{1}_{t-l} \\ &+ \frac{r}{2}st\delta_{t+l}\delta_{s+k}(b,v)(a,u)\mathbb{1}_{s-k} + \frac{r}{2}st\delta_{s+l}\delta_{t+k}(a,v)(b,u)\mathbb{1}_{t-k}, \end{aligned} \quad (10)$$

and \mathcal{L} is also related to \mathcal{L}_r . We will discuss this again in Section 5. It's easy to see that $\bar{V}_{\mathcal{J},r}$ in case 1 and 2 can also be uniformly written as:

$$\bar{V}_{\mathcal{J},r} = \text{span}\{L_{a_1,b_1}^r(-m_1,-n_1)\cdots L_{a_k,b_k}^r(-m_k,-n_k)\cdot 1 \mid a_i,b_i \in \mathfrak{h}, m_i, n_i \geq 1\}. \quad (11)$$

with the central charge equals to dr .

4 Dual Pair Realization of $\bar{V}_{\mathcal{J},r}$, $r = -2n+1, n \geq 1$

In this section we give detailed constructions of Case 3. This case is slightly different from Case 1 and 2 because we need to consider an orthosymplectic superspace W , the corresponding orthosymplectic Lie algebra $\mathfrak{osp}(1|2n)$ and the corresponding orthosymplectic ‘supergroup’ $Osp(1|2n)$ which act on W .

Recall that a superspace W is a $\mathbb{Z}/2\mathbb{Z}$ -graded space with $W = W_{\bar{0}} \oplus W_{\bar{1}}$, where $W_{\bar{0}}$ and $W_{\bar{1}}$ are called even and odd part of W respectively. A superspace W is called orthosymplectic if W has a supersymmetric bilinear form (\cdot, \cdot) , such that (\cdot, \cdot) restricts to $W_{\bar{0}}$ is non-degenerate symmetric, to $W_{\bar{1}}$ is symplectic, and $W_{\bar{0}}, W_{\bar{1}}$ are orthogonal to each other:

$$(u, v) = 0, \forall u \in W_{\bar{0}}, v \in W_{\bar{1}}.$$

For our purpose we set

$$W_{\bar{0}} = V_m, W_{\bar{1}} = W_n.$$

We say the ‘superdimension’ of W is $(m|2n)$ and we write:

$$\text{sdim}(W) = (m|2n).$$

Given an orthosymplectic super space W with $\text{sdim}(W) = (m|2n)$ we have the corresponding Lie superalgebra $\mathfrak{osp}(m|2n)$ and the corresponding ‘supergroup’ $Osp(m|2n)$. For general theory about $\mathfrak{osp}(m|2n)$ and $Osp(m|2n)$, see for example, [Kac77],[Ser01],[DKW⁺99]. The orthosymplectic ‘supergroup’ $Osp(m|2n)$ here means the ‘super Harich-Chandra pair’ $(\mathfrak{osp}(m|2n), O(m) \times Sp(2n))$ (See for example [DKW⁺99]) where $O(m) \times Sp(2n)$ acts on $\mathfrak{osp}(m|2n)$ through the adjoint action:

$$g \cdot x \stackrel{\text{def.}}{=} gxg^{-1}.$$

We say $Osp(m|2n)$ ‘acts’ on a superspace M , which means that M is a $(\mathfrak{osp}(m|2n), O(m) \times Sp(2n))$ -module such that:

$$g(xv) = (g \cdot x)(gv), \quad \forall g \in O(m) \times Sp(2n), x \in \mathfrak{osp}(m|2n), v \in M.$$

It’s easy to see that $O(m) \times Sp(2n)$ acts on $M^{\mathfrak{osp}(m|2n)}$ so we define:

$$M^{Osp(m|2n)} \stackrel{\text{def.}}{=} (M^{\mathfrak{osp}(m|2n)})^{O(m) \times Sp(2n)}.$$

Construction of Case 3, $r = -2n + 1, n \geq 1$. We now focus on the special case $\text{sdim}(W) = (1|2n)$. We simply say ‘a superspace W ’ and we omit the adjective ‘orthosymplectic’ for convenience. Observe that $\mathfrak{h} \otimes W$ is a superspace with the supersymmetric bilinear form:

$$(a \otimes u, b \otimes v) = (a, b)(u, v), \quad \forall a \otimes u, b \otimes v \in \mathfrak{h} \otimes W.$$

The even and odd parts are given by

$$(\mathfrak{h} \otimes W)_{\bar{0}} = \mathfrak{h} \otimes V_1 \simeq \mathfrak{h}, \quad (\mathfrak{h} \otimes W)_{\bar{1}} = \mathfrak{h} \otimes W_n.$$

It is easy to see that we have a corresponding super Lie algebra $\widehat{\mathfrak{h} \otimes W}$:

$$\widehat{\mathfrak{h} \otimes W} = (\mathfrak{h} \otimes W) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

with the super Lie bracket given by:

$$[a(m), b(n)] = m(a, b)\delta_{m+n, 0}c, \quad [x, c] = 0, \quad \forall x \in \widehat{\mathfrak{h} \otimes W}.$$

Here $a(m) = at^m$. It is also easy to check that

$$(\widehat{\mathfrak{h} \otimes W})_- \stackrel{\text{def.}}{=} (\mathfrak{h} \otimes W) \otimes \mathbb{C}t^{-1}[t^{-1}]$$

is a super-commutative Lie subalgebra. The corresponding SVOA is essentially isomorphic to a tensor product SVOA:

$$\mathcal{H}(\mathfrak{h} \otimes V_1) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n) \simeq \mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n).$$

It's obvious that $Osp(d|2nd)$ acts on $\widehat{\mathfrak{h} \otimes W}$ and $\mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n)$. $Osp(1|2n)$ acts on the component W , and we define $\bar{V}_{\mathcal{J}, -2n+1}$ as:

$$\bar{V}_{\mathcal{J}, -2n+1} \stackrel{def.}{=} (\mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n))^{Osp(1|2n)}.$$

There is also an invariant theory for orthosymplectic supergroups. Set

$$L_{a,b}^{-2n+1}(k, l) \stackrel{def.}{=} L_{a,b}^1(k, l) + L_{a,b}^{-2n}(k, l).$$

By the invariant theory for $Osp(1|2n)$ [Ser01], [LZ16], [LZ14] we have a fixpoint Lie subalgebra \mathcal{L}_{-2n+1} :

$$\mathcal{L}_{-2n+1} \stackrel{def.}{=} (\widehat{\mathfrak{h} \otimes W})^{Osp(1|2n)} = \text{span}\{L_{a,b}^{-2n+1}(k, l), c \mid a, b \in \mathfrak{h}, k, l \in \mathbb{Z}\}.$$

Note that our observation at the end of Section 4 also applies here. The formula (10) and (11) still hold in this case. The Virasoro element is given by:

$$\omega = \sum_k L_{e_k, e_k}^{-2n+1}$$

and the central charge equals to $dr = -(2n-1)d$.

So our construction of $\bar{V}_{\mathcal{J}, r}$ in all three cases can be unified using the approach of this section, and (10) gives the commutation relation for all cases. Case 1(Case 2, Case 3, resp.) corresponds to construction using superspace with superdimension $(m|0)$ ($(0|2n)$, $(1|2n)$, resp.).

We remark that these dual-pair constructions are analogues to dual pairs $(O(m), C_\infty)$ ($(Sp(2n), C_\infty)$, $(Osp(1|2n), C_\infty)$, resp.) studied by W. Wang in [Wan99a], [Wan99b]. For each finite dimensional simple module for $O(m)$, $(Sp(2n)$, $Osp(1|2n)$, resp.) there is a corresponding C_∞ -module which is also the corresponding (simple) $\bar{V}_{\mathcal{J}, r}$ -module, where r is the level with respect to each case.

5 Properties of $\bar{V}_{\mathcal{J}, r}$, $r \in \mathbb{Z}_{\neq 0}$

In this section we prove Theorem 1 and we fix the following notation:

$$V = \bar{V}_{\mathcal{J}, r} = \text{span}\{L_{a_1, b_1}^r(-m_1, -n_1) \cdots L_{a_k, b_k}^r(-m_k, -n_k) \cdot 1 \mid a_i, b_i \in \mathfrak{h}, m_i, n_i \geq 1\}.$$

The fomrula (10) will be important for our computations.

Proof of (1) in Theorem 1. The central charges have already been computed in Section 3 and 4 so we check the isomorphism between the Griess algebra V_2 and the Jordan algebra \mathcal{J} here. First $V_0 = \mathbb{C}1$, $V_1 = \{0\}$ is obvious, and

$$V_2 = \text{span}\{L_{a,b}^r(-1, -1) \cdot 1 \mid a, b \in \mathfrak{h}\}$$

is also clear. By (10) it is computed that

$$L_{a,b}^r(1)L_{u,v}^r = L_{a,b}^r(-1, 1)L_{u,v}^r(-1, -1) \cdot 1 + L_{a,b}^r(1, -1)L_{u,v}^r(-1, -1) \cdot 1$$

$$=(b, u)L_{a, v}^r + (b, v)L_{a, u}^r + (a, u)L_{b, v}^r + (a, v)L_{b, u}^r.$$

So

$$L_{a, b}^r \mapsto L_{a, b}, V_2 \rightarrow \mathcal{J}$$

gives the isomorphism.

Proof of (2) in Theorem 1. The proof is essentially similar to the proof of Proposition 3.1 in [NS10]. It is enough to prove this for $d = 2$. We may assume that a, b form an orthonormal basis of \mathfrak{h} such that:

$$(a, a) = (b, b) = 1, (a, b) = 0.$$

Let \bar{M}_r denote the VOA which is generated by V_2 :

$$\bar{M}_r = \text{span}\{L_{a_1, b_1}^r(l_1) \cdots L_{a_k, b_k}^r(l_k) \cdot 1 \mid a_i, b_i \in \mathfrak{h}, l_i \in \mathbb{Z}\}.$$

We need to show $V = \bar{M}_r$. $\bar{M}_r \subseteq V$ is obvious, and it is enough to prove the converse.

We prove it by induction on the ‘length’ of elements in V . For

$$L_{a_1, b_1}^r(-m_1, -n_1) \cdots L_{a_k, b_k}^r(-m_k, -n_k) \cdot 1 \in V,$$

we call it is of ‘length k ’. We use $P(k)$ to denote the subspace of V spanned by elements of length less or equal to k . So we have a filtration:

$$\mathbb{C} \cdot 1 = P(0) \subseteq \cdots P(k) \subseteq \cdots V.$$

When $k = 0$, $\mathbb{C} \cdot 1 \in \bar{M}_r$ obviously holds.

Suppose $P(k) \in \bar{M}_r$ already holds. Let:

$$x \stackrel{\text{def}}{=} L_{a_1, b_1}^r(-m_1, -n_1) \cdots L_{a_k, b_k}^r(-m_k, -n_k).$$

We want to show

$$L_{a, a}^r(k, l)x \cdot 1, L_{a, b}^r(k, l)x \cdot 1 \in \bar{M}_r. \quad (12)$$

First we prove

Lemma 4.

$$L_{a, b}^r(-1, -1)x \cdot 1 \in \bar{M}_r.$$

Proof of Lemma 4. Let $y \stackrel{\text{def}}{=} \sum_{k \neq -1} L_{a, b}^r(-k - 2, k)$. We check that

$$L_{a, b}^r(-1) = L_{a, b}^r(-1, -1) + \sum_{k \neq -1} L_{a, b}^r(-k - 2, k) = L_{a, b}^r(-1, -1) + y$$

and

$$L_{a, b}^r(m, n) \cdot 1 = 0, [L_{a, b}^r(m, n), x] \cdot 1 \in P(k), \text{ if } m \geq 0, \text{ or } n \geq 0$$

hold. We also check that $L_{a,b}^r(-1)x \cdot 1 \in \bar{M}_r$ and:

$$\begin{aligned} L_{a,b}^r(-1)x \cdot 1 &= L_{a,b}^r(-1, -1)x \cdot 1 + yx \cdot 1 \\ &= L_{a,b}^r(-1, -1)x \cdot 1 + [y, x] \cdot 1. \end{aligned}$$

So

$$L_{a,b}^r(-1, -1)x \cdot 1 = L_{a,b}^r(-1)x \cdot 1 - [y, x] \cdot 1 \in \bar{M}_r.$$

and we conclude the proof of Lemma 4.

For the remaining part we divide (12) into two cases:

Case 1. $L_{a,b}^r(-m, -n)x \cdot 1 \in P(k+1) \subseteq \bar{M}_r, \forall m, n \geq 1.$

Case 2. $L_{a,a}^r(-m, -n)x \cdot 1 \in P(k+1) \subseteq \bar{M}_r, \forall m, n \geq 1.$

Proof of Case 1. Take two 'Virasoro like' elements $L_{a,a}^r, L_{b,b}^r$. A direct computation shows:

$$\begin{aligned} [L_{a,a}^r(0), L_{a,b}(-m, -n)] &= mL_{a,b}^r(-m-1, -n) \\ [L_{b,b}^r(0), L_{a,b}(-m, -n)] &= nL_{a,b}^r(-m, -n-1). \end{aligned}$$

So we have:

$$\begin{aligned} &\frac{L_{a,a}^r(0)^{m-1}L_{b,b}^r(0)^{n-1}}{(m-1)!(n-1)!}L_{a,b}^r(-1, -1)x \cdot 1 \\ &= L_{a,b}^r(-m, -n)x \cdot 1 + L_{a,b}^r(-1, -1)\frac{L_{a,a}^r(0)^{m-1}L_{b,b}^r(0)^{n-1}}{(m-1)!(n-1)!}x \cdot 1 \in \bar{M}_r \end{aligned}$$

by induction hypothesis. Note that

$$\begin{aligned} &L_{a,b}^r(-1, -1)\frac{L_{a,a}^r(0)^{m-1}L_{b,b}^r(0)^{n-1}}{(m-1)!(n-1)!}x \cdot 1 \\ &= L_{a,b}^r(-1, -1)\left[\frac{L_{a,a}^r(0)^{m-1}L_{b,b}^r(0)^{n-1}}{(m-1)!(n-1)!}, x\right] \cdot 1 \in \bar{M}_r. \end{aligned}$$

By Lemma 4 we have

$$L_{a,b}^r(-m, -n)x \cdot 1 \in \bar{M}_r.$$

So we conclude the proof of this case.

Proof of Case 2. First it is shown that

$$\begin{aligned} &(L_{a,b}^r(-m, -n) \cdot 1)(k)x \cdot 1 \\ &= \left[\frac{L_{a,a}^r(0)^{m-1}L_{b,b}^r(0)^{n-1}}{(m-1)!(n-1)!}, L_{a,b}^r(k)\right]x \cdot 1 \in \bar{M}_r. \end{aligned}$$

From the proof of Case 1 we have

$$L_{b,a}^r(1)L_{a,b}^r(-m, -n)x \cdot 1 \in \bar{M}_r,$$

$$\begin{aligned}
& (L_{b,a}^r(-2, -1) \cdot 1)(2)L_{a,b}^r(-m, -n)x \cdot 1 \in \bar{M}_r, \\
& L_{a,b}^r(-m, -n)(L_{b,a}^r(-1, -1) \cdot 1)(1)x \cdot 1 \in \bar{M}_r \\
& L_{a,b}^r(-m, -n)(L_{b,a}^r(-2, -1) \cdot 1)(2)x \cdot 1 \in \bar{M}_r.
\end{aligned}$$

Then we deduce that

$$\begin{aligned}
& [(L_{b,a}^r(-1, -1) \cdot 1)(1), L_{a,b}^r(-m, -n)]x \cdot 1 \in \bar{M}_r \\
& [(L_{b,a}^r(-2, -1) \cdot 1)(2), L_{a,b}^r(-m, -n)]x \cdot 1 \in \bar{M}_r.
\end{aligned}$$

A direct computation shows that

$$\begin{aligned}
& [(L_{b,a}^r(-1, -1) \cdot 1)(1), L_{a,b}^r(-m, -n)] = mL_{b,b}^r(-m, -n) \cdot 1 + nL_{a,a}^r(-m, -n) \\
& [(L_{b,a}^r(-2, -1) \cdot 1)(2), L_{a,b}^r(-m, -n)] = m(m-1)L_{b,b}^r(-m, -n) \cdot 1 - n(n+1)L_{a,a}^r(-m, -n).
\end{aligned}$$

Then we have:

$$\begin{aligned}
& mL_{b,b}^r(-m, -n)x \cdot 1 + nL_{a,a}^r(-m, -n)x \cdot 1 \\
& = [(L_{b,a}^r(-1, -1)x \cdot 1)(1), L_{a,b}^r(-m, -n)]x \cdot 1 \in \bar{M}_r \\
& m(m-1)L_{b,b}^r(-m, -n) \cdot 1 - n(n+1)L_{a,a}^r(-m, -n)x \cdot 1 \\
& = [(L_{b,a}^r(-2, -1) \cdot 1)(2), L_{a,b}^r(-m, -n)]x \cdot 1 \in \bar{M}_r.
\end{aligned}$$

Solving this we have

$$L_{b,b}^r(-m, -n)x \cdot 1, L_{a,a}^r(-m, -n)x \cdot 1 \in \bar{M}_r.$$

So we've proved Case 1 and Case 2 for (12). By induction on k , $P(k) \subseteq \bar{M}_r \quad \forall k \geq 1$ so $V \subseteq \bar{M}_r$ and we conclude the proof.

Remark. We note that (1) in Theorem 1 is also satisfied by $V_{\mathcal{J},r}$ (See [AM09]), and (2) in Theorem 1 also holds for $V_{\mathcal{J},r}$ with assumption $d \geq 2$ [NS10]. But (2) fails if $d = 1$ for $V_{\mathcal{J},r}$. Let $\mathcal{H}(\mathfrak{h})^+$ be the fixpoint subVOA of $\mathcal{H}(\mathfrak{h})$ under the action of -1 on \mathfrak{h} . It was shown in [DN99] that when $d = 1$, $\mathcal{H}(\mathfrak{h})^+$ can be generated by Virasoro element ω and another degree 4 element J . This suggests that from the view of Griess algebra, we should exclude the case $d = 1$.

Proof of (3) in Theorem 1. This is done by establishing the relation between \mathcal{L} and \mathcal{L}_r . By (10), it is obvious that the following map

$$U(\mathcal{L})/(c-r) \rightarrow U(\mathcal{L}_r)/(c-1), \quad L_{a,b}(m, n) \mapsto L_{a,b}^r(m, n) \quad (13)$$

is an associative algebra homomorphism. Here $(c-r)$ and $(c-1)$ means the corresponding two sided ideals generated by $c-r$ and $c-1$ respectively. Note that $L_{a,b}(k), L_{a,b}^r(k)$ are in certain completion of $U(\mathcal{L})$ and $U(\mathcal{L}_r)$ respectively. By the remark at the end of the proof to (2) of Theorem 1, $V_{\mathcal{J},r}$ is generated by $(V_{\mathcal{J},r})_2$ when $d \geq 2$, so the map (13) naturally extends to a VOA homomorphism

$$V_{\mathcal{J},r} \rightarrow \bar{V}_{\mathcal{J},r} \quad (14)$$

The surjectivity of this map follows from (2) of Theorem 1 that V is generated by V_2 when $d \geq 2$.

We have the following lemma:

Lemma 5 ([DLM96], Theorem 2.4 and Theorem 2.8). *If V is a simple VOA and the action of a Lie algebra \mathfrak{g} on V is semisimple, then $V^{\mathfrak{g}}$ is also simple. The same holds if we replace \mathfrak{g} with a group G .*

Let W be an orthosymplectic superspace with $\text{sdim}(W) = (m|0)((0|2n), (1|2n)$ resp.) By using the supersymmetric bilinear form over W , it is easy to check that Fock spaces $\mathcal{H}(\mathfrak{h} \otimes V_m)(\mathcal{A}(\mathfrak{h} \otimes W_n), \mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n)$ resp.) are simple as $\widehat{\mathfrak{h} \otimes V_m}(\widehat{\mathfrak{h} \otimes W_n}, \widehat{\mathfrak{h} \otimes W}, \text{resp.})$ -modules because the induced invariant bilinear form over the corresponding Fock spaces are non-degenerate (For arguments see the proof of Proposition 2.2 in [KR87]). This implies that $\mathcal{H}(\mathfrak{h} \otimes V_m)(\mathcal{A}(\mathfrak{h} \otimes W_n), \mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n), \text{resp.})$ are all simple (S)VOAs.

For Case 1 and 2, it is well known that $O(m), Sp(2n)$ -action are semisimple so by Lemma 5, $\bar{V}_{\mathcal{J},r}$ is simple when $r = m \geq 1$ or $r = -2n, n \geq 1$. For Case 3 because

$$\begin{aligned} \bar{V}_{\mathcal{J},-2n+1} &= (\mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n))^{Osp(1|2n)} \\ &= ((\mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n))^{\mathfrak{osp}(1|2n)})^{O(1) \times Sp(2n)}. \end{aligned}$$

and we note the following lemma:

Lemma 6 (See for example, [Sch79], p.239, Theorem 1). *The category of the finite dimensional $\mathfrak{osp}(1|2n)$ -module is semisimple if and only if $\text{sdim}(W) = (m|0), (0|2n), \text{ or } (1|2n)$.*

So by Lemma 6, $M = \mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n)$ is decomposed into a direct sum of irreducible $\mathfrak{osp}(1|2n)$ -modules:

$$M = M^{\mathfrak{osp}(1|2n)} \bigoplus \bigoplus M^\lambda,$$

where M^λ are non-trivial irreducible $\mathfrak{osp}(1|2n)$ -modules. Note that the non-degenerated bilinear form over M is invariant under the $\mathfrak{osp}(1|2n)$ -action, thus $M^{\mathfrak{osp}(1|2n)}$ is orthogonal to all M^λ and we deduce that the invariant bilinear form restricted to $M^{\mathfrak{osp}(1|2n)}$ is non-degenerate. By similar arguments applied to $G = O(1) \times Sp(2n) \simeq \{\pm 1 \times Sp(2n)\}$ and $V = M^{\mathfrak{osp}(1|2n)}$, we deduce that the invariant bilinear form restricted to $\bar{V}_{\mathcal{J},-2n+1} = V^G$ is also non-degenerated, thus $\bar{V}_{\mathcal{J},-2n+1}$ is a simple VOA [Li94]. Therefore we have:

Proposition 1. $\bar{V}_{\mathcal{J},r}(r \in \mathbb{Z}_{\neq 0})$ are all simple.

So we conclude the proof of (3) in Theorem 1 as a corollary of this proposition. We remark that by the second fundamental theorem of invariants for orthosymplectic supergroup [LZ16][LZ14], it is easy to see that the kernel of the map (13) is non-zero. We can write down some elements in the kernel explicitly, and this explains Proposition 6.1 in [NS10] about ‘high symmetry of singular vectors’. As another corollary, we also reprove that $V_{\mathcal{J},r}$ is reducible when $r \in \mathbb{Z}$ (The case when $r = 0$ is trivial).

6 Character Formulas of Simple Quotient $\bar{V}_{\mathcal{J},r}, r = -2n, n \geq 1$

In this section we give character formulas for $\bar{V}_{\mathcal{J},r}$ in Case 2, $r = -2n, n \geq 1$. By Theorem 1,

$$\bar{V}_{\mathcal{J},-2n} = \mathcal{A}(\mathfrak{h} \otimes W_n)^{Sp(2n)}.$$

Because $Sp(2n)$ is simply connected so it is enough to calculate the character of $\mathcal{A}(\mathfrak{h} \otimes W_n)^{sp(2n)}$ here. We set $\mathfrak{g} = \mathfrak{sp}(2n)$, and $V = \bar{V}_{\mathcal{J},-2n}$ in this section.

It is known from Section 3 that the Virasoro element $\omega \in V$ given by

$$\omega = \sum_k L_{e_k, e_k}^{-2n},$$

and $L(0) = \omega(1)$ gives the $\mathbb{Z}_{\geq 0}$ -grading on V :

$$V = \bigoplus_{i \geq 0} V_i, \quad V_i = \{v \in V \mid L(0)v = iv\}.$$

Let \mathfrak{g}_0 be the Cartan subalgebra of \mathfrak{g} . It is easy to check that \mathfrak{g} -action commutes with $L(0)$, so $\mathcal{A}(\mathfrak{h} \otimes W_n)$ is decomposed into common eigenspaces of \mathfrak{g}_0 and $L(0)$ labeled by a pair $(\alpha, k), \alpha \in (\mathfrak{g}_0)^*, k \in \mathbb{Z}_{\geq 0}$:

$$\mathcal{A}(\mathfrak{h} \otimes W_n) = \bigoplus_{(\alpha, k)} \mathcal{A}(\mathfrak{h} \otimes W_n)(\alpha, k).$$

Define the q -graded formal character $\text{ch}_q(\mathcal{A}(\mathfrak{h} \otimes W_n))$ to be:

$$\text{ch}_q(\mathcal{A}(\mathfrak{h} \otimes W_n)) \stackrel{\text{def.}}{=} \sum_{(\alpha, k)} \dim(\mathcal{A}(\mathfrak{h} \otimes W_n)(\alpha, k)) e^\alpha q^k.$$

Note that

$$\mathcal{A}(\mathfrak{h} \otimes W_n) = \bigwedge (\widehat{\mathfrak{h} \otimes W_-}).$$

So

$$\text{ch}_q(\mathcal{A}(\mathfrak{h} \otimes W_n)) = \prod_{i=1, \dots, d, j \geq 1} (1 + e^{-\epsilon_i} q^j)^d (1 + e^{\epsilon_i} q^j)^d. \quad (15)$$

In particular when $d = 1$ we have:

$$\text{ch}_q(\mathcal{A}(W_n)) = \prod_{i=1, \dots, d, j \geq 1} (1 + e^{-\epsilon_i} q^j)(1 + e^{\epsilon_i} q^j). \quad (16)$$

On the other hand, the \mathfrak{g} -action on the Fock space $\mathcal{A}(\mathfrak{h} \otimes W_n)$ is semisimple. Because all finite dimensional simple \mathfrak{g} -modules are isomorphic to $L(\lambda)$ for some $\lambda \in \Lambda_+^0$. Then we have a decomposition:

$$\mathcal{A}(\mathfrak{h} \otimes W_n) = \bigoplus_{\lambda \in \Lambda_+^0} (L(\lambda) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n)^{L(\lambda)}),$$

where $\mathcal{A}(\mathfrak{h} \otimes W_n)^{L(\lambda)}$ is the ‘multiplicity space’ with respect to $L(\lambda)$ on which $L(0)$ acts. In particular $\mathcal{A}(\mathfrak{h} \otimes W_n)^{\mathfrak{g}}$ is the multiplicity space with respect to the trivial representation $\lambda = 0$. By using this decomposition we have:

$$\text{ch}_q(\mathcal{A}(\mathfrak{h} \otimes W_n)) \stackrel{\text{def.}}{=} \sum_{\lambda \in \Lambda_+^0} \text{ch}(L(\lambda)) \text{Tr}|_{\mathcal{A}(\mathfrak{h} \otimes W_n)^{L(\lambda)}} q^{L(0)}. \quad (17)$$

In particular when $d = 1$ we have:

$$\text{ch}_q(\mathcal{A}(W_n)) \stackrel{\text{def.}}{=} \sum_{\lambda \in \Lambda_+^0} \text{ch}(L(\lambda)) \text{Tr}|_{\mathcal{A}(W_n)^{L(\lambda)}} q^{L(0)}.$$

Following the notation in [CL16], we define the ‘branching functions’ $B_\lambda(q)$:

$$B_\lambda(q) \stackrel{\text{def.}}{=} \text{Tr}|_{\mathcal{A}(W_n)^{L(\lambda)}} q^{L(0)}.$$

and we rewrite the above as:

$$\text{ch}_q(\mathcal{A}(W_n)) \stackrel{\text{def.}}{=} \sum_{\lambda \in \Lambda_+^0} \text{ch}_q(L(\lambda)) B_\lambda(q). \quad (18)$$

We remark that the ‘character’ in [CL16] means $\text{Tr} q^{L(0) - \frac{c}{24}}$ so our definitions of ‘branching functions’ are slightly different.

The explicit formula for $B_\lambda(q)$ has been obtained by Linshaw and Creutzig in [CL16], as Corollary 5.5. They derive it by applying the Jacobi triple product identity to (16) and compare it with (18). Introduce an element:

$$\rho \stackrel{\text{def.}}{=} \rho_0 - \rho_1.$$

Let W^0 denote the Weyl group of $\mathfrak{sp}(2n)$. Then their formula reads:

$$B_\lambda(q) = P(q)^n \sum_{w \in W^0} (-1)^{l(w)} q^{\frac{1}{2}(w(\lambda + \rho_0) - \rho, w(\lambda + \rho_0) - \rho) - \frac{1}{2}(\rho_1, \rho_1)}. \quad (19)$$

Note that the element ρ is exactly the half sum of positive roots in Φ , and the Weyl group of $\mathfrak{so}(2n + 1)$ is isomorphic to W^0 , then we can rewrite (19) the same as (1). Define the ‘specialization of type λ ’ F_λ on the formal character by:

$$F_\lambda(e^\mu) = e^{(\lambda, \mu)}.$$

We have

$$\begin{aligned} B_\lambda(q) &= P(q)^n q^{\frac{1}{2}(w(\lambda + \rho_0), w(\lambda + \rho_0)) + \frac{1}{2}(\rho, \rho) - \frac{1}{2}(\rho_1, \rho_1)} \sum_{w \in W^0} (-1)^{l(w)} q^{-(\lambda + \rho_0, w(\rho))} \\ &= P(q)^n q^{\frac{1}{2}(\lambda + \rho_0, \lambda + \rho_0) + \frac{1}{2}(\rho, \rho) - \frac{1}{2}(\rho_1, \rho_1)} F_{-\lambda - \rho_0} \left(\sum_{w \in W^0} (-1)^{l(w)} e^{w(\rho)} \right) \\ &= P(q)^n q^{\frac{1}{2}(\lambda + \rho_0, \lambda + \rho_0) + \frac{1}{2}(\rho, \rho) - \frac{1}{2}(\rho_1, \rho_1)} F_{-\lambda - \rho_0} \left(e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \right) \end{aligned}$$

$$= q^{\frac{1}{2}(\lambda+\rho_1, \lambda+\rho_1) - \frac{1}{2}(\rho_1, \rho_1)} P(q)^n \prod_{\alpha \in \Phi^+} (1 - q^{(\lambda+\rho_0, \alpha)}).$$

Here we use the denominator identity of $\mathfrak{so}(2n+1)$:

$$e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = \sum_{w \in W^0} (-1)^{l(w)} e^{w(\rho)},$$

and we note that the inner product (\cdot, \cdot) is invariant under the Weyl group action. The formula (1) when $\lambda = 0$ is obtained in [CL16], as Corollary 5.6.

Combine (15), (16), and (18) together we have:

$$\begin{aligned} \text{ch}_q(\mathcal{A}(\mathfrak{h} \otimes W_n)) &= (\text{ch}_q(\mathcal{A}(W))^d \\ &= \left(\sum_{\lambda \in \Lambda_+^0} \text{ch}(L(\lambda)) B_\lambda(q) \right)^d \\ &= \sum_{\lambda_1, \dots, \lambda_d \in \Lambda_+^0} \text{ch}(L(\lambda_1)) \cdots \text{ch}(L(\lambda_d)) B_{\lambda_1}(q) \cdots B_{\lambda_d}(q) \\ &= \sum_{\mu \in \Lambda_+^0} m_{\lambda_1, \dots, \lambda_d}^\mu \text{ch}(L(\mu)) B_{\lambda_1}(q) \cdots B_{\lambda_d}(q). \end{aligned}$$

Compare this with (17) and use the fact that $\text{ch}(L(\lambda))$ are linearly independent, we have:

$$\text{Tr}|_{\mathcal{A}(\mathfrak{h} \otimes W_n)^{L(\mu)}} q^{L(0)} = \sum_{\mu \in \Lambda_+^0} m_{\lambda_1, \dots, \lambda_d}^\mu B_{\lambda_1}(q) \cdots B_{\lambda_d}(q).$$

So Theorem 2 is obtained by taking $\mu = 0$. We remark that $m_{\lambda, \mu}^\nu$ are called Clebsch-Gordan coefficients of $\mathfrak{sp}(2n)$, and $m_{\lambda_1, \dots, \lambda_d}^\mu$ can be expressed by $m_{\lambda, \mu}^\nu$. It's an interesting fact that our character formula is related to these Clebsch-Gordan coefficients.

References

- [AM09] T. Ashihara and M. Miyamoto. Deformation of central charges, vertex operator algebras whose Griess algebras are Jordan algebras. *Journal of Algebra*, 321(6):1593–1599, 2009.
- [Ash11] T. Ashihara. On a VOA Associated with the simple Jordan algebra of type D . *Communications in Algebra*, 39(6):2097–2113, 2011.
- [CL16] T. Creutzig and A. Linshaw. Orbifolds of symplectic fermion algebras. *Transactions of the American Mathematical Society*, 2016.
- [DKW⁺99] P. Deligne, D. Kazhdan, E. Witten, L. Jeffrey, D. Freed, P. Etingof, D. Morrison, and J. Morgan. Quantum fields and string: a course for mathematicians, Vol 1. 1999.

- [DLM96] C. Dong, H. Li, and G. Mason. Compact automorphism groups of vertex operator algebras. *International Mathematics Research Notices*, 1996(18):913–921, 1996.
- [DN99] C. Dong and K. Nagatomo. Classification of irreducible modules for the vertex operator algebra $M(1)^+$. *Journal of Algebra*, 216(1):384–404, 1999.
- [FLM89] I. Frenkel, J. Lepowsky, and A. Meurman. *Vertex operator algebras and the monster*, volume 134. Academic Press, 1989.
- [Hum08] J. Humphreys. *Representations of semisimple Lie algebras in the BGG category O* , volume 94. American Mathematical Soc., 2008.
- [JJ49] F. Jacobson and N. Jacobson. Classification and representation of semi-simple Jordan algebras. *Transactions of the American Mathematical Society*, 65(2):141–169, 1949.
- [Kac77] V. Kac. A sketch of lie superalgebra theory. *Communications in Mathematical Physics*, 53(1):31–64, 1977.
- [Kac94] V. Kac. *Infinite-dimensional Lie algebras*, volume 44. Cambridge university press, 1994.
- [Kac98] V. Kac. *Vertex algebras for beginners*. Number 10. American Mathematical Soc., 1998.
- [Kau95] G. Kausch. Curiosities at $c=-2$. *arXiv preprint hep-th/9510149*, 1995.
- [KR87] V. Kac and A. Raina. *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*. World Scientific, 1987.
- [KR93] V. Kac and A. Radul. Quasifinite highest weight modules over the lie algebra of differential operators on the circle. *Communications in mathematical physics*, 157(3):429–457, 1993.
- [Lam96] C. Lam. Construction of vertex operator algebras from commutative associative algebras. *Communications in Algebra*, 24(14):4339–4360, 1996.
- [Lam99] C. Lam. On VOA associated with special Jordan algebras. *Communications in Algebra*, 27(4):1665–1681, 1999.
- [Li94] H. Li. Symmetric invariant bilinear forms on vertex operator algebras. *Journal of Pure and Applied Algebra*, 96(3):279–297, 1994.
- [LZ14] G. Lehrer and R. Zhang. The second fundamental theorem of invariant theory for the orthosymplectic supergroup. *arXiv preprint arXiv:1407.1058*, 2014.

- [LZ16] G. Lehrer and R. Zhang. The first fundamental theorem of invariant theory for the orthosymplectic supergroup. *Communications in Mathematical Physics*, pages 1–42, 2016.
- [NS10] H. Niibori and D. Sagaki. Simplicity of a vertex operator algebra whose Griess algebra is the Jordan algebra of symmetric matrices. *Communications in Algebra*, 38(3):848–875, 2010.
- [Sch79] M. Scheunert. *The theory of Lie superalgebras, Lecture Notes in Mathematics 716*. 1979.
- [Ser01] A. Sergeev. An analog of the classical invariant theory for Lie superalgebras I. *The Michigan Mathematical Journal*, 49(1):113–146, 2001.
- [Wan99a] W. Wang. Dual pairs and infinite dimensional Lie algebras. *Contemporary Mathematics*, 248:453–469, 1999.
- [Wan99b] W. Wang. Duality in infinite dimensional Fock representations. *Communications in Contemporary Mathematics*, 1(02):155–199, 1999.

DEPARTMENT OF MATHEMATICS, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, KOWLOON

Email Address: **hzhaoab@ust.hk**